Further Calculus with Inverse Hyperbolic functions (A Level Only)

Inverse hyperbolic functions can be differentiated implicitly. The results can be used to solve some more difficult integration problems.

Implicit Differentiation of Inverse Hyperbolic Functions

Example 1: Given that $y = \text{arsinh}(x)$, find $\frac{dy}{dx}$ in terms of x.

First, rearrange $y = \text{arsinh}(x)$ to find x in terms of y. This is found by taking sinh of both sides of the equation.	This means that $y = \text{arsinh}(x)$ $x = \sinh(y)$
Differentiate this with respect to y.	$\frac{dx}{dy} = \cosh(y)$
As $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, find $\frac{dy}{dx}$ by taking the reciprocal of both sides.	$\frac{dy}{dx} = \frac{1}{\cosh(y)}$
The question asks for the derivative to be given in terms of x. Given that $x = \sinh(y)$, the derivative needs to be rewritten in terms of $\sinh(y)$. To do this, use the identity $\cosh^2(y) - \sinh^2(y) \equiv 1$. Since $\cosh(y) \geq 1$, only the positive square root is needed here.	$\cosh^2(y) - \sinh^2(y) = 1$ $\cosh^2(y) = \sinh^2(y) + 1$ $\cosh(y) = \sqrt{\sinh^2(y) + 1}$
Substitute this back into the expression for the derivative.	$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{\sinh^2(y) + 1}}$
Rewrite in terms of x to give the result.	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$

The results for $\text{arcosh}(x)$ and $\text{artanh}(x)$ can be found in the same way. The following are provided in the formula book:

$$\frac{d}{dx}\text{arsinh}(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\text{arcosh}(x) = \frac{1}{\sqrt{x^2-1}}, x > 1$$

$$\frac{d}{dx}\text{artanh}(x) = \frac{1}{1-x^2}, |x| < 1$$

Integration Techniques Using Inverse Hyperbolic Functions

The results above can be used along with algebraic manipulation to solve some trickier integrals. The derivative of $\text{artanh}(x)$ can be split into partial fractions which integrates to produce a result involving logarithms. Integrals in the form of the derivatives of $\text{arsinh}(x)$ and $\text{arcosh}(x)$ can be solved by use of an appropriate substitution. The standard results that are given in the formula book are:

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \text{arsinh}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \text{arcosh}\left(\frac{x}{a}\right) + c$$

Example 2: Use the substitution $x = a\cosh(u)$ to prove the result $\int \frac{1}{\sqrt{x^2-a^2}}dx = \text{arcosh}\left(\frac{x}{a}\right) + c$.

First, differentiate the substitution to express dx in terms of du.	$x = a\cosh(u)$ $\frac{dx}{du} = a\sinh(u)$ $\therefore dx = a\sinh(u)\,du$
Rewrite the integral in terms of the substitution, expressing the integrand in terms of u.	$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{a^2\cosh^2(u) - a^2}} a\sinh(u)\,du$ $= \int \frac{1}{\sqrt{a^2(\cosh^2(u) - 1)}} a\sinh(u)\,du$
Use the identity $\cosh^2(u) - \sinh^2(u) \equiv 1$ to simplify the integrand.	$\cosh^2(u) - \sinh^2(u) = 1$ $\sinh^2(u) = \cosh^2(u) - 1,$ $= \int \frac{1}{\sqrt{a^2(\cosh^2(u) - 1)}} a\sinh(u)\,du$ $= \int \frac{1}{\sqrt{a^2\sinh^2(u)}} a\sinh(u)\,du$
Now the integral reads:	$= \int \frac{1}{a\sinh(u)} a\sinh(u)\,du$ $\int 1\,du = u + c$
Substitute $u = \text{arcosh}\left(\frac{x}{a}\right)$ back in to finish the proof.	$u + c = \text{arcosh}\left(\frac{x}{a}\right) + c$

In some cases, the integral may not appear to be in the form of an inverse hyperbolic and may require algebraic manipulation to be written in the form of these standard results. This can be done by factorising the integrand or by splitting it into partial fractions.

Example 3: Find $\int_6^{10} \frac{1}{\sqrt{(x^2-4x-12)}}dx$ in exact form.

First, complete the square of the quadratic in the denominator.	$x^2 - 4x - 12$ $= (x-2)^2 - 4 - 12$ $= (x-2)^2 - 16$
Rewrite the integral with the factorised denominator.	$\int \frac{1}{\sqrt{x^2-4x-12}}dx = \int \frac{1}{\sqrt{(x-2)^2 - 16}}dx$
This is now in the form $\frac{1}{\sqrt{u^2-a^2}}$ with $u = x - 2$ and $a = 4$, so the integral is in the form for the result using $\text{arcosh}\left(\frac{u}{a}\right)$ to be used.	$\int_6^{10} \frac{1}{\sqrt{(x-2)^2 - 4^2}} = \left[\text{arcosh}\left(\frac{x-2}{4}\right)\right]_6^{10}$ $= \left(\text{arcosh}\left(\frac{8}{4}\right) - \text{arcosh}\left(\frac{4}{4}\right)\right)$ $= \text{arcosh}(2) - \text{arcosh}(1)$
Use the logarithmic form of $\text{arcosh}(x)$ to find the exact answer.	$= \ln\left(2 + \sqrt{2^2 - 1}\right) - \ln\left(1 + \sqrt{1^2 - 1}\right)$ $= \ln(2 + \sqrt{3}) - \ln(1)$ $= \ln(2 + \sqrt{3})$